

Lance 3 - Projections & Unitaries

Let A be a C^* -algebra; E, F are Hilbert C^* -modules

Recall $L(E, F) = \{t: E \rightarrow F \mid \exists t^*: F \rightarrow E \text{ st } \underbrace{\langle tx, y \rangle = \langle x, t^*y \rangle}_{\in A} \forall x \in E, y \in F\}$

$L(E)$ is a unital C^* -algebra

Goal: pursue $L(E, F) \sim B(\mathcal{H}_1, \mathcal{H}_2)$

In $B(\mathcal{H})$

- p is a projection if $p^2 = p = p^*$

$\Leftrightarrow p$ projects \mathcal{H} onto a closed subspace K

Note $\mathcal{H} = K \oplus K^\perp$

- u is a unitary if $u^*u = uu^* = 1$ ($u^{-1} = u^*$)

$\Leftrightarrow u$ is a surjective isometry

- w is an isometry $w^*w = 1 \Leftrightarrow \|wx\| = \|x\| \forall x \in \mathcal{H}$

For now $F \subseteq E$ is a closed module

Def F is ^(orthogonally) complemented if $F \oplus F^\perp = E$

Facts: • If $F \subseteq E$ is complemented $\forall z \in E \ z = x + y \in F \oplus F^\perp$

then $p: E \rightarrow E \ z \mapsto x$ is a projection in $L(E)$

- If $p \in L(E)$ is a projection, then $\text{ran}(p)$ is complemented with $\text{ran}(p)^\perp = \text{ran}(1-p) = \ker(p)$

Def $F \subseteq E$ is topologically complemented if $\exists G \subseteq E$ a closed submodule st $F \oplus G = E \quad F \cap G = \{0\}$

Ex | Let $A = C([0, 1])$, $J = \{f \in A \mid f(0) = 0\} \subseteq A$

Let $E = A \oplus J$ and $F = \{(f, f) \mid f \in J\}$

Then $F^\perp = \{(g, -g) \mid g \in J\}$ $\langle (f, f), (g, -g) \rangle = \langle f, g \rangle + \langle f, -g \rangle$

But $F \oplus F^\perp = J \oplus J \neq E$

So F is not complemented

F is top. complemented with $G = \{(g, 0) \mid g \in A\}$

Define $q: E \rightarrow E$ by $(f, g) \mapsto (f - g, 0)$

q is bounded, $q^2 = q$, $\text{ran}(q) = G$ is complemented

$G^\perp = \{(0, g) \mid g \in J\}$ but $q \notin L(E)$

F is a Hilbert A -module

Thm | Suppose $t \in L(E, F)$ has closed range. Then

1) $\ker(t)$ is complemented

2) $\text{ran}(t)$ " "

3) $\text{ran}(t^*)$ is closed

Ex | Let $A = C([-1, 1])$, $F = E = A$

Define $t: A \rightarrow A$ by $t(f(\lambda)) = \begin{cases} \lambda f(\lambda) & 0 \leq \lambda \leq 1 \\ 0 & -1 \leq \lambda \leq 0 \end{cases}$

$\ker(t) = \{f \in A \mid f(\lambda) = 0 \quad 0 \leq \lambda \leq 1\}$

$\text{ran}(t) = \{f \in A \mid f(\lambda) = 0 \quad -1 \leq \lambda \leq 0\}$

Neither of these are complemented.

Def $u \in L(E, F)$ is a unitary if $u^*u = I_E$ and $uu^* = I_F$

Thm Let $u: E \rightarrow F$ be linear. TFAE

① u is isometric ($\|ux\| = \|x\|$), A -linear, surjective.

② u is a unitary in $L(E, F)$

Lemma: Let $a, b \in A_+$. If $\|ac\| = \|bc\| \forall c \in A$ then $a = b$.

Pf: ② \Rightarrow ① u is a unitary $\Rightarrow u$ is surjective, A -linear

u is isometric since

$$\|ux\|_F^2 = \|\langle ux, ux \rangle\|_A = \|\langle x, u^*ux \rangle\|_A = \|\langle x, x \rangle\|_A = \|x\|_E^2 \quad \checkmark$$

① \Rightarrow ② Let $x \in E, a \in A$

Recall $|x| := \langle x, x \rangle^{1/2} \geq 0$

$$\begin{aligned} \|ux\|_F \|a\| &= \|\langle ux, ux \rangle^{1/2} a\| \\ &= \|a^* \langle ux, ux \rangle a\|^{1/2} \\ &= \|\langle u(xa), u(xa) \rangle\|^{1/2} \\ &= \|u(xa)\| \\ &= \|xa\| \\ &= \| |x| a \| \end{aligned}$$

$$\text{Lemma} \Rightarrow \|ux\| = \|x\| \Rightarrow \langle ux, ux \rangle = \langle x, x \rangle$$

$$\text{Polarization} \Rightarrow \langle ux, uy \rangle = \langle x, y \rangle \quad \forall y \in E$$

$$u \text{ is bijective} \Rightarrow u^{-1} = u^*$$

$w: E \rightarrow F$ isometric, A -linear $\nRightarrow w^*w = I_E$

Ex $A = C([0, 1])$, $J = \{f \in A \mid f(0) = 0\}$

Then $i: \mathcal{J} \hookrightarrow A$ is A -linear, isometric, not adjointable.

Thm | Let $\omega: E \rightarrow F$ be linear. TFAE

- ① ω A -linear, isometric, with complemented range
- ② $\omega \in L(E, F)$ and $\omega^* \omega = I_E$

Def | IF \exists a unitary $u: E \rightarrow F$ then E and F are unitarily equivalent, denoted $E \approx F$

Ex | Let \mathcal{H} be a Hilbert space with ONB $\{e_i\}$
 $\mathcal{H} \otimes A \approx \bigoplus_i A$ via $e_i \otimes a \mapsto (\delta_{ij} a)_j$

Ex | Let X be a locally compact Hausdorff space.
 $\mathcal{H} \otimes C_0(X) \approx C_0(X, \mathcal{H})$ via $h \otimes f \mapsto (x \mapsto f(x)h)$

Polar decomposition

- In $B(\mathcal{H})$: $t = v|t|$ where v is a partial isometry
 $\Leftrightarrow v = vv^*v$

Lemma | For $t \in L(E, F)$, $\overline{\text{ran}(t^*)} = \overline{\text{ran}(t^*t)}$

Thm | IF $\exists t \in L(E, F)$ such that t, t^* both have dense range,
then $E \approx F$

"pf": By lemma $\overline{\text{ran}(t^*t)} = E$. So the operator
defined by $|t| := (t^*t)^{\frac{1}{2}}$ also has dense range.
Then $E \approx F$ via $u: \text{ran}(t) \rightarrow \text{ran}(|t|)$ $tx \mapsto |t|x$ " " "

Def $v \in L(E, F)$ is a partial isometry (from $E_0 \subseteq E$ to $F_0 \subseteq F$)
 if $F_0 := \text{ran}(v)$ is complemented and if $\exists E_0 \subseteq E$ a
 complemented submodule st $v|_{E_0}: E_0 \rightarrow F_0$ is a unitary
 and $v(E_0^\perp) = \{0\}$.

Facts: 1) $v \in L(E, F)$ is a partial isometry $\iff v = vv^*v$
 2) IF $t \in L(E, F)$ has $\overline{\text{ran}(t)}$ and $\overline{\text{ran}(t^*)}$ are both
 complemented then t has polar decomposition
 $t = v|t|$ where v is a partial isometry